L.23
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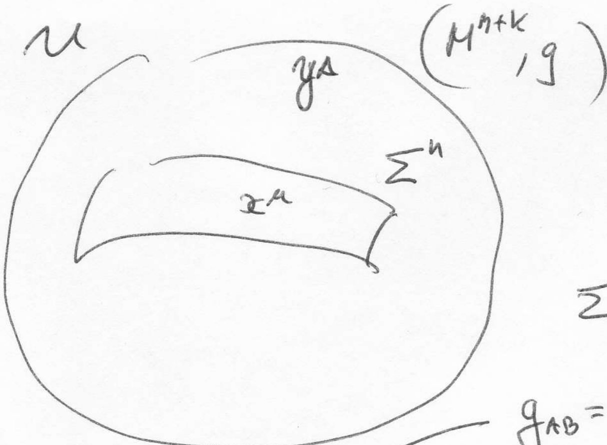
Local isometric embedding in (M^{n+k}, g).



y^A - local coordinates in $U \subset M^{n+k}$

$A = 1, 2, \dots 4, \dots n+k$

$$g = g_{AB}\, dy^A dy^B \qquad g_{AB} = g_{AB}(y^C)$$

$$\Sigma^n = \{ y^A = y^A(x^\mu),\ dx^1 \wedge \dots \wedge dx^n \neq 0 \}$$

$g_{AB} = \ddot{g}_{AB}(x^\alpha)$

$$g|_\Sigma = \underbrace{g_{AB}\, y^A{}_{,\mu}\, y^B{}_{,\nu}}_{\tilde{g}_{\mu\nu}}\, dx^\mu dx^\nu = \tilde{g}_{\mu\nu}\, dx^\mu dx^\nu = \delta_{\mu\nu}\, \theta^\mu \theta^\nu$$

orthonormal frame.

$(\theta^1, \dots \theta^n)$ orthonormal frame on Σ

$(e_1, \dots e_n)$ dual frame.

$$dy^A = e_\mu(y^A)\theta^\mu = e_\mu{}^A \theta^\mu$$

$e_\mu(y^A) = e_\mu^A$ n-vectors at each point $x \in \Sigma^n$

These vectors have values at $T_x M^{n+k}$

$$\left. \begin{array}{l} g_{AB}\, dy^A dy^B|_\Sigma = e_\mu^A e_\nu^B \theta^\mu \theta^\nu g_{AB} \\ \quad \| \\ \delta_{\mu\nu} \theta^\mu \theta^\nu \end{array} \right\} \Rightarrow \boxed{g_{AB}\, e_\mu^A e_\nu^B = \delta_{\mu\nu}}$$

or:

$$g(e_\mu, e_\nu) = \delta_{\mu\nu}$$

We supplement e_μ^A by n_a^A ~~...~~ so that (e_μ^A, n_a^B) is an orthonormal basis in $T_x M^{n+k}$ at each point $x \in \Sigma^n$.

We again have:

$$\boxed{\begin{cases} de_\mu{}^A = b_{\mu a}\, n_a{}^A + \gamma_{\nu\mu}\, e_\nu{}^A \\ dn_b{}^A = \alpha_{ab}\, n_a{}^A + \beta_{b\mu}\, e_\mu{}^A \end{cases} \quad \text{and} \quad dy^A = e_\mu{}^A \theta^\mu .}$$

We have to supplement this by the compatibility conditions $d^2 = 0$.

We in addition have:

1) $\boxed{d\theta^\mu + \Gamma_{\mu\nu} \wedge \theta^\nu = 0}$ and $\Gamma_{\mu\nu} = -\Gamma_{\nu\mu}$ are Levi-Civita connection 1-forms for $g|_\Sigma$ in frame θ^μ.

2)
$$\left.\begin{aligned} &Dg_{AB} = dg_{AB} - \Gamma_{AB} - \Gamma_{BA} \\ &\text{and} \\ &d(dy^A) + \Gamma^A{}_B \wedge dy^B = 0 \end{aligned}\right\} \text{ in } \underline{M}$$

$$\Rightarrow \quad \left.\begin{aligned} &dg_{AB} = \Gamma_{AB} + \Gamma_{BA} \\ &\Gamma^A{}_B \wedge dy^B = 0 \end{aligned}\right\} \text{ in } M$$

restricting to Σ we have:

$$dg_{AB} = \left(\Gamma_{AB\rho} + \Gamma_{BA\rho}\right) \wedge \theta^\rho$$

$$e^B_\mu\, \Gamma^A{}_{B\rho}\, \theta^\rho \wedge \theta^\mu = 0$$

$$\Rightarrow \quad \boxed{\Gamma^A{}_{B\rho}\, e^B_\mu = \Gamma^A{}_{B\mu}\, e^B_\rho} \quad \boxed{dg_{AB} = \left(\Gamma_{AB\rho} + \Gamma_{BA\rho}\right) \wedge \theta^\rho}$$

3) $$\begin{cases} g(e_\mu, e_\nu) = \delta_{\mu\nu} = g_{AB}\, e_\mu{}^A e_\nu{}^B \\ g(e_\mu, n_a) = 0 = g_{AB}\, e_\mu{}^A n_a{}^B \\ g(n_a, n_b) = 0 = g_{AB}\, n_a{}^A n_b{}^B \end{cases}$$

$$\begin{cases} b_{\mu a} = de_\mu^A\, n_a^B\, g_{AB} = g(de_\mu, n_a) \\ \gamma_{\nu\mu} = de_\mu^A\, e_\nu^B\, g_{AB} = g(de_\mu, e_\nu) \\ \alpha_{ab} = dn_b^A\, n_a^B\, g_{AB} = g(dn_b, n_a) \\ \beta_{a\nu} = dn_a^A\, e_\nu^B\, g_{AB} = g(dn_a, e_\nu) \end{cases}$$

Closing the system:

$$0 = d^2 y^A = \left(b_{\mu a} n^A{}_a + \gamma_{\nu\mu} e_\nu{}^A\right) \wedge \theta^\mu + e_\mu{}^A \left(-\Gamma_{\mu\nu} \wedge \theta^\nu\right)$$

$$\Rightarrow \boxed{\begin{array}{l} b_{\mu a} \wedge \theta^\mu = 0 \\ (\gamma_{\nu\mu} - \Gamma_{\nu\mu}) \wedge \theta^\nu = 0 \end{array}}$$

Since $b_{\mu a} = b_{\mu\nu a}\theta^\nu \Rightarrow b_{\mu\nu a}\,\theta^\nu \wedge \theta^\mu = 0 \Rightarrow b_{\mu\nu a} = b_{\nu\mu a}$

$\Rightarrow \boxed{b_a = b_{\mu\nu a}\,\theta^\mu \theta^\nu}$ second fundamental form.

What we know about $\gamma_{\nu\mu}$?

$$0 = d\delta_{\mu\nu} = dg_{AB}\, e_\mu^A e_\nu^B + g_{AB}\, de_\mu^A e_\nu^B + g_{AB}\, e_\mu^A de_\nu^B =$$

$$= dg_{AB}\, e_\mu^A e_\nu^B + \gamma_{\nu\mu} + \gamma_{\mu\nu}$$

$$dg_{AB}\, e_\mu{}^A e_\nu{}^B = (\Gamma_{AB\rho} + \Gamma_{BA\rho}) e^A{}_\mu e^B{}_\nu \theta^\rho$$

$$dg_{AB}\, e^A{}_\nu e^B{}_\mu = (\Gamma_{AB\rho} + \Gamma_{BA\rho}) e^A{}_\nu e^B{}_\mu \theta^\rho$$